Math 1552

Section 8.8:

Improper Integrals (cont.)

Math 1552 lecture slides adapted from the course materials
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Example B.1: Evaluate the following integral: $\int_0^1 \frac{\ln(x)}{\sqrt{x}}dx = I$

$\rightarrow$ problem at $x=0$ (vertical asymptote):

$$\lim_{x\to 0^+} \frac{\ln(x)}{\sqrt{x}} = -\infty$$

$$\rightarrow I = \lim_{a\to 0^+} \int_a^1 \frac{\ln(x)}{\sqrt{x}} dx \quad (*)$$

$\rightarrow$ evaluate the indefinite integral first:

$$\int \frac{\ln(x)}{\sqrt{x}}\,dx = 2\cdot\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx$$

$$\left(\text{s-sub: } s=\sqrt{x},\ ds = \frac{dx}{2\sqrt{x}}\right)$$

$$= 4\cdot\int \ln(s)\,ds$$

$$\left(\begin{array}{lll} \text{IBP:} & u = \ln(s) & dv = ds \\ & du = \frac{ds}{s} & v = s \end{array}\right)$$

$$= 4\left(s\ln(s) - \int ds\right)$$

$$= 4\left(s\ln(s) - s\right) + C$$

$$= 2\sqrt{x}\cdot\ln(x) - 4\sqrt{x} + C$$

$\longrightarrow$ we will need

$$\lim_{x \to 0^+} \sqrt{x}\cdot\ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{\sqrt{x}}} \quad \frac{\infty}{\infty}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{2}\right) \cdot \frac{1}{x^{3/2}}} = -2 \lim_{x \to 0^+} \sqrt{x}$$

$$= -2 \quad (\cancel{**})$$

$\longrightarrow$ back to $(\cancel{*})$:

$$I = \lim_{a \to 0^+} \left(2\sqrt{x}\ln(x) - 4\sqrt{x} \right) \Big|_a^1$$

$$= \left(2\sqrt{1}\cdot \ln(1) - 4\sqrt{1}\right) - \lim_{a\to 0^+}\left(2\sqrt{a}\ln(a) - 4\sqrt{a}\right)$$

$$= -4 - \left(2(-2) - 4\cdot 0\right), \text{ by } (**)$$

$$= 4$$

Example B.2: Evaluate the following integral: $\int_0^\infty \frac{e^{-\frac{1}{2x}}}{x^2}\,dx = I$

$\rightarrow$ problem at $x=0$ (vertical asymptote)

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{2x}}}{x^2} = +\infty$$

$\rightarrow$ also, the upper limit of integration is infinite

$$\rightarrow I = \lim_{\substack{a\to 0^+ \\ b\to\infty}} \int_a^b \frac{e^{-\frac{1}{2x}}}{x^2}\,dx \quad (*)$$

→ evaluate the indefinite integral:

$\left(u\text{-sub: } u = \frac{1}{2x}, \ du = -\frac{dx}{2x^2} \right)$

$$\int \frac{e^{-\frac{1}{2x}}}{x^2}\,dx = -2\int e^{-u}\,du$$

$$= 2e^{-u} + C$$

$$= 2e^{-\frac{1}{2x}} + C$$

$\rightarrow$ back to $(*)$:

$$I = \lim_{b\to\infty} 2e^{-\frac{1}{2b}} - \lim_{a\to 0^+} 2e^{-\frac{1}{2a}}$$

$$= \lim_{x\to 0} 2e^{-x} - \lim_{x\to\infty} 2e^{-x}$$

$$= 2 - 0 = 2$$

Example B.3: Evaluate the following integral: $\int_0^\infty \frac{e^x}{e^{2x}+3}dx = I$

→ No vertical asymptotes, but the upper limit of integration is infinite

→ $I = \lim_{b\to\infty} \int_0^b \frac{e^x}{e^{2x}+3}dx \quad (*)$

→ evaluate the indefinite integral:

$(u\text{-sub: } u = e^x,\ du = e^x dx)$

$$\int \frac{e^x}{e^{2x}+3}\,dx = \int \frac{du}{u^2+3}$$

$$= \frac{1}{3}\int \frac{du}{\left(\frac{u}{\sqrt{3}}\right)^2+1} \quad \left(v\text{-sub: } v = \frac{u}{\sqrt{3}},\ dv = \frac{du}{\sqrt{3}}\right)$$

$$= \frac{\sqrt{3}}{3}\int \frac{dv}{1+v^2} = \frac{\sqrt{3}}{3}\tan^{-1}(v) + C$$

$$= \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{e^x}{\sqrt{3}}\right) + C$$

$\rightarrow$ back to $(*)$:

$$I = \lim_{b \to \infty} \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{e^b}{\sqrt{3}}\right) - \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \lim_{x\to\infty} \frac{\sqrt{3}}{3}\tan^{-1}(x) - \frac{\sqrt{3}}{3}\cdot\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{3}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi\sqrt{3}}{9}$$

Example B.4: Evaluate the following integral: $\int_1^\infty \frac{dx}{x\sqrt{\ln(x)}} = I$

$\rightarrow$ problem at $x=1$ (vertical asymptote)

$$\lim_{x\to 1} \frac{1}{\sqrt{\ln(x)}} = +\infty$$

$\rightarrow$ also, the upper limit of integration is infinite

$$\rightarrow I = \lim_{\substack{a\to 1^+ \\ b\to\infty}} \int_a^b \frac{dx}{x\sqrt{\ln(x)}} \quad (*)$$

$\longrightarrow$ evaluate the indefinite integral:

$\left(u\text{-sub}: \; u = \ln(x), \; du = \dfrac{dx}{x} \right)$

$$\int \frac{dx}{x\sqrt{\ln(x)}} = \int u^{-\frac{1}{2}}\, du = 2\sqrt{u} + C$$

$$= 2\sqrt{\ln(x)} + C$$

$\longrightarrow$ back to $(*)$:

$$I = \lim_{b\to\infty} 2\sqrt{\ln(b)} - \lim_{a\to 1^+} 2\sqrt{\ln(a)}$$

$$= +\infty - 2\sqrt{0} = +\infty$$

(The improper integral diverges)